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The Observer-dependence of Cellular Space-time Structure

E. A. B. COLE

School of Mathematics, University of Leeds, Leeds LS2 9JT

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Abstract

It is argued that space-time has a cellular structure, the exact structure being observer-dependent and consistent with the amount of energy he has available for refining his measuring apparatus. The usual concept of a single distance in continuous space is replaced by the concept of a distance set between cells, the elements of each set depending on the cellular structures of both the space and the measuring rod that is used in the measurement. The idea that there are many different ways of measuring the same observable is abandoned: instead, the definition of the original observable becomes split by the different measuring processes used, and the results of a measurement of each new observable defined by this splitting are predicted from the eigenvalues of a common operator by using an observer-dependent construction. Transformations between observers with different cellular structures are considered. The transformation is not as exact as in the continuous case, with at best a cell of one space being associated with a set of cells in the other. This transformation is determined by information being exchanged by the observers concerning the locations in their two spaces of a finite number of common events. The transformation becomes more exact as more information is exchanged.

1. *Introduction*

In a recent paper (Bohm *et al.* 1970) the view was expressed that it is not sufficient to merely change the formulation of physical theory, it is also necessary to change the informal language which describes the physical phenomena. In their words, 'In any new approach that is relevant to physics, discreteness should appear as a natural consequence of the informal considerations and should not be arbitrarily imposed. Thus it is not possible to obtain discreteness naturally if the classical notions of particle, trajectory, potential, field, etc., continue to be taken as primitive concepts since these notions were developed specifically for the continuum.' This would seem to be a correct view of the situation. For example, suppose we wish to use the idea of a cellular space in our theory by introducing fundamental lengths into our existing equations. Such a procedure would be inconsistent if the equations contained the concepts of distance, velocity, etc., because these concepts are not defined in a cellular space. We must either abandon these

equations, or else make new interpretations of these concepts by regarding them as non-observable. New observable quantities would then have to be found—quantities which depend on the complete system-apparatus complex, because there can be no separation between these two.

However, the author disagrees with Hiley & Stuart (1971) who express the view that a cellular structure of space-time is not supported by direct experimental evidence. For whenever we set up our measuring instruments we must always be content with a certain degree of non-refinement, and whenever we convey information about the positions of events, etc., we always do so in a discrete way. This non-refinement of our apparatus has the effect of imposing a cellular structure on space-time, and measurements refer to each cell as a whole (or set of cells) and not to individual points within the cells. Of course, the cellular structure can be refined by refining the apparatus, but this requires additional cost and (equivalently) additional energy. It is only by supplying an infinite amount of energy to the apparatus that the space-time takes on a continuous structure. (The fact that an infinite amount of energy is needed for this process can be illustrated by many thought-experiments.) With this in mind we may postulate the basic principle of cellular space-time structure:

An observer works within a cellular space-time structure the fineness of which is limited by the amount of energy he has available for refining his apparatus. The observer requires infinite energy in order to be able to set up a continuous space-time structure.

This means that space-time always has a cellular structure, but the exact structure is observer-dependent. In practice this means that an observer must decide how much energy he has available for refining his apparatus, and he must then be content with the fineness of the associated cellular structure.

Since the structure is observer dependent we must allow the observer to take any structure he pleases, consistent with the amount of energy he has at his disposal for refining his apparatus. This means that the structure may be non-uniform. But the one which is easiest to handle is the rectangular structure and this is usually introduced into existing equations by using 'fundamental' parameters ξ_μ ($\mu = 1, 2, 3, 4$). Continuous coordinates are then replaced by $n_\mu \xi_\mu$ where n_μ is the cell number in the μ'th direction:

$$x_\mu \to n_\mu \xi_\mu \tag{1.1}$$

and differential operators must be replaced by central-difference operators (Cole, 1970):

$$\frac{\partial}{\partial x_\mu} \to \frac{E_\mu - E_\mu^{-1}}{2\xi_\mu} \tag{1.2}$$

where E_μ is the operator defined for all functions f by

$$E_\mu f(n_1, \ldots, n_4) = f(n_1, \ldots, n_\mu + 1, \ldots, n_4).$$

Care must be taken in the interpretation of the ξ_μ—it would be inconsistent to regard it as the length of the cell edge in the μ'th direction, because the usual idea of length is not valid in a cellular space. (An alternative concept of length based on a possible measuring process is introduced in Section 2.) Rather, the ξ_μ will be interpreted as parameters which enable the formal substitutions (1.1) and (1.2) to be made. Furthermore the ξ_μ are observer-dependent, and they tend to zero in the continuous limit which is an ideal state brought about by the observer providing infinite energy to refine his apparatus.

Using the transition (1.2) one finds that, using periodic boundary conditions, the eigenvalues of the difference operator corresponding to the momentum operator in the μ'th direction are (Cole, 1970)

$$p_q = \hbar \xi_\mu^{-1} \sin(2\pi q / N_\mu) \tag{1.3}$$

for integers q and fixed integer N_μ so that the magnitudes of the eigenvalues are less than or equal to $\hbar \xi_\mu^{-1}$. It is well known (Ingraham, 1967) that a high momentum cut-off enables one to get rid of the divergences which are encountered in the renormalisation theory which is constructed within a continuous space-time structure. By invoking the above postulate it would not be inconsistent to say that this divergence is merely a reappearance of the infinite energy that had to be put in at the beginning to make the theory continuous. As mentioned before, however, the concept of momentum is not defined in a cellular space, so that the eigenvalues (1.3) can no longer be regarded as observable quantities. But it may still be possible to use these eigenvalues to calculate actual observed values, the way in which we use these eigenvalues depending on the way the observation is made. The general eigenvalue problem is tackled in Section 3.

Thus our starting point is a cellular space-time whose structure is observer-dependent. An observer is able to say in which cell (or set of cells) an event is located, but he is unable to further localise the event at a specific point within the cell. Two main problems arising are (i) to define new observable quantities to replace the concepts of the length, momentum etc., and (ii) to find how two or more observers with different cellular structures can communicate the results of observations to one another. This second problem is tackled in Section 4.

2. *The Distance Sets*

From the considerations of the last section we see that we must consider the relationships between cells rather than between individual points within the cells. The concept of distance is based upon the extension between individual points, and so this concept must be modified in a cellular theory in such a way that it is based on a real measuring process. The conventional definition of the distance between two subsets A and B of a pseudo-metric space is

$$\text{dist}(A, B) = \inf\{\text{dist}(x, y) : x \in A \text{ and } y \in B\}$$

This definition as well as the Hausdorff metric, although convenient for many purposes, is arbitrary and not based on the distance-measuring processes which observers actually use.

We must first decide what this distance-measuring operation is and then see what quantities we get from such an operation. Suppose that the 'distance' is to be measured between the three-dimensional cells specified by the integers n_1 and n_2. We use a measuring rod r which itself has a cellular structure consisting of a long string of non-overlapping three-dimensional cells. We then concentrate on two points, e_1 in n_1 and e_2 in n_2, and lay the rod so that e_1 and e_2 lie in some cells of the rod. By noting these cells, the distance between e_1 and e_2 is read off the scale according to some convention. In general it will be possible to move the rod such that although e_1 stays in the same cell of the rod, the point e_2 will appear in a different cell, so that a new distance is read off the scale. If the rod has a non-uniform cellular structure it will be possible to translate the rod so that many more readings of the scale will correspond to e_1 and e_2. By taking all possible pairs of such points in n_1 and n_2 we then generate a set of distances, each element in the set corresponding to a reading of the rod when it is laid against the cells n_1 and n_2. We will denote the distance set by $d_r(n_1,n_2)$; the elements of d will depend on the cellular structures of both the space and the rod and the number of elements of $d_r(n_1,n_2)$ will depend not only on r but also on n_1 and n_2. Note that although we have done something that the observer is not allowed to do, namely split up the cells into individual points, by eventually removing the framework of individual points we arrive at a measuring process which is not incompatible with the cellular structure. That is, we tabulate the incidences between n_1 and n_2 and the cells of the measuring rod.

Clearly such a process can also be extended to measuring the distance sets between different regions L and M formed from sets of cells. Strictly speaking the sets $d_r(L,M)$ should be defined between regions of space, which are themselves cells in a coarser structure, and not between sets of cells, but to simplify the notation we will make no distinction between these so that L, etc. will denote both a set of cells and the region formed from that set.

Having obtained a set of incidence between the cells n_1 and n_2 and the cells of the rod, the observer is at liberty to construct his distance set in any manner he pleases. But in what follows we will consider only those constructions which give the following properties to the d-sets: for all sets of cells L, M and N,

(a) all elements of $d_r(L,M)$ are real, non-negative and distinct;
(b) $d_r(L,M) = d_r(M,L)$;
(c) $0 \in d_r(L,L)$;
(d) $\min d_r(L,N) \leqslant \max d_r(L,M) + \max d_r(M,N)$;
(e) $$d_r(L,M) = \bigcup_{l\in L} \bigcup_{m\in M} d_r(l,m) \tag{2.1}$$

These properties are chosen so that they tie up with the metric properties of continuous space when the limit is taken. For suppose that we refine our

apparatus so that we get a finer cellular structure which is obtained by a subdivision of our original structure. Let n_{1i} and n_{2i} be cells obtained by i subdivisions of cells n_1 and n_2 respectively. The transition to the continuous limit is accomplished by letting the number of cells in a given section of the measuring rod become infinite and by letting i tend to infinity. We should then expect that all the elements of $d_r(n_{1i}, n_{2i})$ converge to one value which will be the usual distance between the two points to which n_{1i} and n_{2i} converge.

In the special case in which the rod has a uniform cubic structure specified by the parameter ξ the distance set $d_r(n_1, n_2)$ could be taken to have the form

$$d_r(n_1, n_2) = \{K\xi : K \text{integer}, K_1(n_1, n_2) \leqslant K \leqslant K_2(n_1, n_2)\}$$

In particular suppose that the cellular space has a cubic structure also specified by the parameter ξ. Then if we specify each cell by three coordinate numbers with $n \equiv (n_1, n_2, n_3)$ and $m \equiv (m_1, m_2, m_3)$ we can take

$$d_r(n, m) = \left\{K\xi : K \text{integer} = \sqrt{\left[\sum_{i=1}^{3} (|n_i - m_i| + \epsilon_i)^2 - \alpha_2{}^2 - \alpha_3{}^2\right]} - \alpha_1,\right.$$

$$\left. -1 < \epsilon_i < 1, -1 < \alpha_i < 1\right\}$$

For example, if $m = (0,0,0)$ and $n = (10,0,0)$ then $d_r(m,n) = \{8\xi, 9\xi, 10\xi, 11\xi, 12\xi\}$, and for all l, $d_r(l,l) = \{0, \xi, 2\xi\}$.

3. *The Interpretation of Eigenvalues*

A broadly held tacit assumption made in physics is that there are many equivalent ways of measuring a given observable, and this assumption is very useful because it greatly simplifies the theories we use to describe physical processes. For example, in quantum mechanics we postulate an operator corresponding to an observable such that its eigenvalues correspond to the quantities which are observed, with no mention of the actual measuring process involved. In this viewpoint we regard the observable as a thing in itself with the measuring process added as an afterthought. However, if we agree with Bohm *et al.* (1970) that, in rough terms, we should define the observable by the way in which it is measured, then we should drop the above assumption. In particular this would mean that the eigenvalues predicted by quantum mechanics are no longer the exact quantities that are actually measured, so that we must either give a new interpretation of the eigenvalues, or reject them altogether as a means of predicting measurements.

It may be possible to use these sets of eigenvalues by making the following interpretation: each operator in quantum mechanics corresponds to a class of measuring processes, each process defining its own observable. Whereas using the assumption all processes in a class define the 'same' observable,

without the assumption we now have a splitting of this definition. The possible set of values which is measured in each process is then predicted by constructing it from the set of common eigenvalues, and this construction process will depend on the individual observing process.

To illustrate how this construction process might work we will consider the case in which the result of a measurement is recorded as a line on a sheet of paper which forms part of the measuring apparatus. The paper will have some cellular structure so that from the distance sets corresponding to this line and to previously known values of the observable, the observed quantity can be calculated. The calculating will be done both for a linear and a quadratic scale. We will assume that both processes correspond to the same class of measuring processes characterised by the operator A with eigenvalues a_l. The operator A itself may depend on the cellular structure imposed by the apparatus and in that case so will the a_l (for example, see the momentum eigenvalues (1.3)). In the continuous limit these eigenvalues will tend to the usual eigenvalues denoted by a_l'.

Linear scale. The measuring instrument records the observation as a straight line on a sheet of paper which has some cellular structure, and there will be a distance set associated with the cells in which the end points of the line lie. Previously a 'known' value a of the observable was recorded as corresponding to the distance set s. If a further measurement is now made, and it is found that a distance set d corresponds to the line obtained, then assuming a linear scale we may associate a set of observed values

$$\{d_i\, a/s_j : d_i \in d,\ s_j \in s\} \tag{3.1}$$

with the distance set d. In the continuous limit all the elements of d and all the elements of s will converge to their respective limits, so that the elements of (3.1) will all converge to one value.

The set (3.1) is obtained by measurement and we must now try to predict those values from the eigenvalues a_l of the quantum mechanical operator. Now if we use a measuring rod with a cubic structure specified by the parameter ξ, then the elements of s will have the form

$$(m + \epsilon_1)\,\xi \qquad (\epsilon_1 = 0, 1, 2, \ldots \mu_1)$$

for some integer m, and the elements of d will have the form

$$(n + \epsilon_2)\,\xi \qquad (\epsilon_2 = 0, 1, 2, \ldots \mu_2)$$

Here a further complication has been added because the values of μ_1 and μ_2 will depend on the cellular structure of the paper. Now consider the set

$$\mathscr{S}(a_l) = \left\{ \frac{[a_l\, m/a] + \epsilon_2}{m + \epsilon_1}\, a : \epsilon_1 = 0, 1, \ldots, \mu_1;\ \epsilon_2 = 0, 1, \ldots, \mu_2 \right\} \tag{3.2}$$

where $[x]$ denotes the integer part of x: this set and the set (3.1) contains the same number of elements, and the elements of both sets have the same

forms, that is, integer $\times a$/interger. Also, by letting $m\xi$ tend to some finite quantity as $\xi \to 0$ it is easy to show that $\mathscr{S}(a_l) \to \{a_l'\}$ in the continuous limit. We then interpret the $\mathscr{S}(a_l)$ by saying that any element of $\bigcup_l \mathscr{S}(a_l)$ is a possible outcome of a measurement of the observable using this linear scale.

There will certainly be other constructions for $\mathscr{S}(a_l)$ which fit the above requirements, but the form (3.2) illustrates the type of construction that will be needed. Note that there is a splitting of the levels a_l, but some of these may be the same as some split levels corresponding to different a_l. We may also find the state of the system from a measurement of the distance set d: if the elements of d are measured to be $n\xi$, $n\xi + \xi$, . . ., $n\xi + \mu_2\xi$ then $[a_l m/a]\xi = n\xi$ so that

$$na/m \leqslant a_l \leqslant (n+1)\,a/m \tag{3.3}$$

Then the system is in any one of the states l such that (3.3) holds.

Quadratic scale. Now the scale is assumed to be quadratic so that it first has to be calibrated by supposing that 'known' values a' and a'' of the observable correspond to the distance sets s' and s'' respectively. If a further observation is made resulting in a distance set d then any one of the elements of the set

$$\left\{\frac{d_i - s_j''}{s_k' - s_j''}a' + \frac{d_i - s_k'}{s_j'' - s_k'}a'' : d_i \in d, s_k' \in s', s_j'' \in s''\right\} \tag{3.4}$$

can be taken to be the value of the observable. Again in the continuous limit all the elements of (3.4) converge to one value.

We must now try to predict the elements of (3.4) using the eigenvalue a_l. If again the measuring rod has a cubic structure specified by the parameter ξ then the elements of d, s' and s'' will have the forms

$$\begin{aligned} &(n+\epsilon)\,\xi && (\epsilon = 0, 1, 2, \ldots, \mu) \\ &(m'+\epsilon')\,\xi && (\epsilon' = 0, 1, 2, \ldots, \mu') \\ &(m''+\epsilon'')\,\xi && (\epsilon'' = 0, 1, 2, \ldots, \mu'') \end{aligned}$$

respectively, where μ, μ' and μ'' are integers which depend on the cellular structure of the paper. Now consider the set

$$\mathscr{S}'(a_l) = \left\{ \frac{\left[\dfrac{a_l(m'-m'') - a''m' + a'm''}{a'-a''}\right] - m'' + \epsilon - \epsilon''}{m'-m''+\epsilon'-\epsilon''}a' + \right.$$

$$\frac{\left[\dfrac{a_l(m'-m'') - a''m' + a'm''}{a'-a''}\right] - m' + \epsilon - \epsilon'}{m''-m'+\epsilon''-\epsilon'}a'' : \epsilon = 0, 1, \ldots, \mu;$$

$$\left.\epsilon' = 0, 1, \ldots, \mu'; \epsilon'' = 0, 1, \ldots, \mu''.\right\} \tag{3.5}$$

this set has the same number of elements as the set (3.4), the elements have the same form and it can be easily shown that $\mathscr{S}'(a_l) \to \{a_l'\}$ in the continuous limit. We then interpret $\mathscr{S}'(a_l)$ by saying that any element of $\bigcup_l \mathscr{S}'(a_l)$ is a possible outcome of a measurement of the observable using this quadratic scale. Again there could be many other constructions with these required properties which make it suitable for this interpretation, but (3.5) illustrates the type of construction that will be needed.

The set (3.5) also gives a splitting of the eigenvalues, but this splitting will be different from the splitting given by (3.2). This distinction does not arise in continuous theory. Thus, the set of predicted quantities which relate to observations on the differing measuring scales will be different, and no meaning can be given to the statement that the observables measured on both instruments are the 'same', although they are equivalent in the sense that both sets of values are constructed from the same set of eigenvalues. Further investigation is needed into the problem of what makes two different observing processes equivalent in this weaker sense.

4. *Transformations Between Observers*

In continuous theory the transformation between the coordinate systems $x_i (i = 1, \ldots, 4)$ and $x_j' (j = 1, \ldots, 4)$ of two observers is given by specifying the function

$$x_j' = f_j(x_1, \ldots, x_4) \qquad (j = 1, \ldots, 4) \tag{4.1}$$

in some region of space-time. If the observers have had no previous contact, the specification can be brought about by an exchange of information about their respective coordinates for each event in the region, and on the basis of this information they then formulate the transformation (4.1). But such a formulation would require an exchange of an infinite amount of information—for example, an infinite amount of information is contained in the statement that the origins are moving with constant relative velocity. Such an exchange would require the expenditure of an infinite amount of energy. Thus the transformation (4.1) represents an ideal situation existing between the observers, one which can never arise using any realistic observation and exchange processes.

In practice, therefore, the observers must be content with a transformation law which is not as detailed as that given in (4.1), with the amount of detail depending on how much information they have exchanged. Furthermore, in a cellular framework the observers must compare coordinates by exchanging information about in which of their cells the common events lie, and will not specify individual points within the cells. This is therefore another cause of lack of detail in the transformation.

Having exchanged some information of this type the next step is to ask which cell, or set of cells R_n of an observer's cellular space-time corresponds to the cell n of the other observer's cellular structure. That is, what is the construction of R_n such that it is possible, consistent with the information

exchanged, for both cell n and any cell of R_n to contain a common event. It is to be expected that as more information is exchanged the sets R_n will contain fewer elements, so that in this sense the transformation becomes more 'exact'. These and other properties will be investigated in the rest of this section. The notation is such that n and n' (with or without suffices) denote individual cells in the cellular spaces S and S' respectively of the two observers, while N, M and N', M', etc., denote sets of cells in these two spaces.

The observers exchange information by specifying to each other the sets of cells in S and S' in which an event occurs, for example the impact of two particles or the explosion of a star. Of course they will have to know that they are both looking at the same event without being able to make reference to their coordinate systems because they are using the event to set up the transformation, but this could always be done with the application of ingenuity.

Suppose the observers look at the events E_α ($\alpha = 1, \ldots, i$), and they each locate the events in their own coordinate systems so that observers with spaces S and S' see E_α located in the sets N_α and N_α' respectively. This information is exchanged between the observers and will be denoted by $I(N_\alpha \leftrightarrow N_\alpha')$. After exchanging i such pieces of information, the total information exchanged will be denoted by

$$I_i \equiv I(N_1 \leftrightarrow N_1', N_2 \leftrightarrow N_2', \ldots, N_i \leftrightarrow N_i')$$

or more shortly by $I(\mathbf{N} \leftrightarrow \mathbf{N}')$. We will assume that none of the pieces of information contradict any other, and in that case $I_j \to I_i$ if $j \geqslant i$. (In the context of this section the symbol $\to$ will mean 'implies'.)

We now introduce the proposition $F_I(N, N')$, defined for all sets N in S and N' in S', which states that consistent with the information I exchanged between the observers it is possible for sets N and N' to contain a common event. Then the following properties are assumed to hold: for each N in S and N' in S',

(i) $F_I(N, N') \to F_J(N, N')$ if $I \to J$;
(ii) for all I consistent with $I(N \leftrightarrow N')$, $F_I(N, N')$ is true;
(iii) $F_I(N, N') \to F_I(M, M')$ for all $M \supset N$, $M' \supset N'$;
(iv) there exist $n \in N$ and $n' \in N'$ such that $F_I(N, N') \to F_I(n, n')$.

Now define

$$R^I_{N;} \equiv \{n' : F_I(N, n') \text{ is true}\}$$
$$R^I_{;N'} \equiv \{n : F_I(n, N') \text{ is true}\} \tag{4.2}$$

that is, $R^I_{N;}$ is the set of cells in S' such that, consistent with the information I, it is possible for any one of these cells to contain an event in common with the region N in S, and $R^I_{;N'}$ is the set of cells in S such that, consistent with the information I, it is possible for any one of these cells to contain an event in common with the region N' in S'. In the continuous theory the transformation (4.1) links a single point in S' with a single point in S, but now in

the cellular theory the best we can do is to link a set of cells $R^I_{n;}$ in S' with the single cell n in S (and $R^I_{;n'}$ gives the reverse transformation from a cell n' in S' to a set of cells $R^I_{;n'}$ in S).

Using the definition (4.2) and the properties (i) to (iv) above, the following results can be found:

(a) For all N and N', J and $I \to J$,

$$\begin{aligned} R^I_{N;} &\subset R^J_{N;} \\ R^I_{;N'} &\subset R^J_{;N'} \end{aligned} \tag{4.3}$$

that is, the greater the amount of information passed between the observers then the smaller the R-sets become. The transmission of further information eliminates some of the possibilities that existed in linking cells in S and S', and the formal transformations

$$n \mapsto R^I_{n;}, \qquad n' \mapsto R^I_{;n'}$$

become more exact. No useful purpose would be served in investigating the quantities $\lim R^I_{n;}$ and $\lim R^I_{;n'}$ as the amount of information I becomes infinite, because the limiting process represents an ideal state which is not allowed to any observer.

(b) For all I and sets N and N',

$$\begin{aligned} \bigcup_{n \in N} R^I_{n;} &= R^I_{N;} \\ \bigcup_{n' \in N'} R^I_{;n'} &= R^I_{;N'} \end{aligned} \tag{4.4}$$

(c) For all I and sets N and N'

$$\begin{aligned} N \subset M &\to R^I_{N;} \subset R^I_{M;} \\ N' \subset M' &\to R^I_{;N'} \subset R^I_{;M'} \end{aligned} \tag{4.5}$$

that is, the smaller the set in S then the smaller is the transform of that set in S', and vice versa.

(d) For all I, $n' \in R^I_{n;}$ if and only if $n \in R^I_{;n'}$ (4.6)

That is, given the transformation from one space to another, result (4.6) is necessary for the consistency of the reverse transformation.

(e) For all I and sets N and N',

$$\begin{aligned} N &\subset R^I_{;R_N;I} \\ N' &\subset R^I_{R^I_{;N';}} \end{aligned} \tag{4.7}$$

Thus starting with set N in S and forming the transformed set $R^I_{N;}$ in S' the set in S corresponding to the reverse transformation of region $R^I_{N;}$ should contain N. This, with the similar result starting with N' in S', further guarantees the consistency of the transformation.

All the above results apply when the cells are either four-dimensional space-time cells or three-dimensional space cells. For the case in which the transformation is between two three-dimensional cellular spaces we will need to include the cellular result equivalent to the preservation of distance by the transformation. This will take the form of a relation between the distance set associated with two cells in one space, and the distance set associated with the corresponding R-sets in the other space.

Let $d_r(m,n)$ denote the distance set obtained with a measuring rod r corresponding to cells m and n in S, and let $d_r'(m',n')$ denote the distance set corresponding to cells m' and n' in S' obtained using the same rod r. Then for a transformation between the two three-dimensional spaces we impose the following limitations on the R-sets, $\varnothing$ denoting the empty set:

(i) given any m and n in S, then for each $m' \in R^{I(\mathrm{N}\leftrightarrow\mathrm{N}')}_{m;}$ and all $n' \in R^{I(\mathrm{N}\leftrightarrow\mathrm{N}',m\leftrightarrow m')}_{n;}$,

$$d_r(m,n) \cap d_r'(m',n') \neq \varnothing$$

(ii) given any m' and n' in S', then for each $m \in R^{I(\mathrm{N}\leftrightarrow\mathrm{N}')}_{;m'}$ and all $n \in R^{I(\mathrm{N}\leftrightarrow\mathrm{N}',m\leftrightarrow m')}_{n';}$,

$$d_r(m,n) \cap d_r'(m',n') \neq \varnothing$$

(iii) given any m in S and any n' in S', then for each $m' \in R^{I(\mathrm{N}\leftrightarrow\mathrm{N}')}_{m;}$ and all $n \in R^{I(\mathrm{N}\leftrightarrow\mathrm{N}'\,m\leftrightarrow m')}_{;n'}$,

$$d_r(m,n) \cap d_r'(m',n') \neq \varnothing$$

Using these conditions together with conditions (2.1) and the results (4.3) one can show that for all non-contradictory I and J,

(iv) for each m and n in S, $d_r(m,n) \cap d_r'(R^I_{m;}, R^J_{n;}) \neq \varnothing$;
(v) for each m' and n' in S', $d_r(R^I_{;m'}, R^J_{;n'}) \cap d_r'(m',n') \neq \varnothing$;
(vi) for each n in S and m' in S', $d_r(R^I_{;m'}, n) \cap d_r'(m', R^J_{n;}) \neq \varnothing$.

Result (iv) shows that the distance set corresponding to m and n in S has at least one element in common with the distance set associated with the corresponding R-sets in S' when the same measuring rod is used. The remaining results are variants of this result. These relations would be much involved if the observers were allowed to use different measuring rods.

To give a simple illustration of the use of these R-sets, suppose that we wish to describe the motion of a particle P in a cellular space. Then the most complete description of the motion is given by specifying the four-dimensional cells through which the particle passes. The *path* of the particle in the space can then be specified by the set N_P of these cells. If N_P and N_P' are the paths of P in the spaces S and S' respectively then these paths must be such that for all I,

$$N_P' \subset R^I_{N_P;} \qquad \text{and} \qquad N_P \subset R^I_{;N_P'}$$

More likely the particle will have extension so that its motion is given by specifying not single cells, but regions $N_P(i)$ and $N_P'(i)$ $(i = 1,2,\ldots)$ through

which the particle passes. The i-dependence of these regions can be taken such that

$$N_P(i) \cap R^I_{;N_{P'}(i)} \neq \varnothing$$

The paths are then given in the spaces by the sets $N_P = \bigcup_i N_P(i)$ and $N_P' = \bigcup_i N_P'(i)$.

Finally, if we are considering the motion of a rigid body and wish to specify as far as possible its linear and rotational motion then we must pick points $P_i (i = 1, \ldots, p)$ of the body and specify the paths of these individual points. If N_{P_i} and N'_{P_i} are these paths in S and S' respectively then the motion in each space is given by specifying the quantities N_{P_1}, N_{P_2}, . . ., N_{P_p} and N'_{P_1}, N'_{P_2}, . . ., N'_{P_p}. As above, we have for all I,

$$N'_{P_i} \subset R^I_{N_{P_i}}; \qquad \text{and} \qquad N_{P_i} \subset R^I_{;N_{P_i}'} \qquad (i = 1, \ldots, p).$$

The larger the value of p then the better is the description of the motion in each space.

Conclusions

The main points which have been made are:

1. The cellular structure of space-time is observer-dependent. The observer is able to set up any structure consistent with the amount of energy he has available for refining his apparatus. Infinite energy is needed to refine the structure to a continuum.

2. The concept of a single distance must be replaced by the concept of a distance set, the elements of which depend on the cellular structures of both the space and the measuring rod used in the measurement.

3. There is a splitting in the usual definition of an observable, and the results of a measurement of each new observable defined by this splitting are predicted from the eigenvalues of a common operator by an observer-dependent construction.

4. The transformation between two cellular structures is not as well defined as in the continuous case, with at best a cell in one space being associated with a set of cells in the other. These sets depend on the information passed between the observers, and the transformation will become more exact, in the sense that these sets will contain fewer elements, as more information is exchanged.

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